Theoretically based expressions in closed form for the local and mean coefficients of skin friction in fully turbulent flow along smooth and rough plates

Stuart W. Churchill
Department of Chemical Engineering, University of Pennsylvania, Philadelphia, PA, USA

The currently accepted correlating equations for the local and mean coefficients of skin friction in the turbulent boundary layer for unconfined flow over smooth and rough plates are wholly empirical. Improved expressions in closed form in which the only empiricism is that arising from the generally accepted semitheoretical expressions for the velocity distribution, including the region of the wake, are derived herein. These new, theoretically based expressions for the coefficients of skin friction are shown by comparison with experimental data to be more accurate for fully turbulent flow ($Re_x > 3 \times 10^6$) than the previous empirical ones based on the same data and/or computed values. Except for possible improvement of the numerical coefficients, these new expressions are not expected to be made obsolete by improved experimental data or numerical solutions.

Keywords: skin friction; drag coefficients; turbulent boundary layer; flat plate; boundary-layer theory

Introduction

Many empirical equations and graphical correlations have been proposed for the coefficients for skin friction as a function of the Reynolds number and roughness ratio for the turbulent regime of flow over smooth and rough plates, respectively. The more relevant ones are examined below in connection with the expressions derived herein. Most of the correlating equations in current use have no theoretical basis for their form. The objective of the work reported herein has been to derive expressions with a theoretically based structure, and to compare them with experimental data and prior expressions.

Although a complete theoretical expression is presumably not attainable for the velocity distribution in a turbulent boundary layer, a number of expressions with a theoretically based structure but empirical coefficients have been devised for different regimes and regions therein. The generally accepted expressions for the friction factor in round tubes (or at least their form) were long ago derived from this fragmentary, semitheoretical structure for the velocity distribution.

The same expressions for the velocity distribution, although with numerically different coefficients in part, are applicable for unconfined flow over a flat plate. However, the derivations and the resulting expressions for the coefficients of skin friction are more complex because (1) the thickness of the unconfined boundary layer is a varying, dependent variable, and (2) the wake at the edge of the boundary layer has a greater influence. The derivations herein nevertheless utilize the above-mentioned semitheoretical structure for the velocity distribution with no further empiricism and with minimal simplifications to produce expressions for the local and mean coefficients of skin friction for smooth and rough plates. Such possibilities were recognized in the past, but rejected because of the anticipated complexity of the resulting expressions. The general availability of hand-held as well as more powerful electronic computers now effectively eliminates that consideration.

Address reprint requests to Professor Churchill at the Department of Chemical Engineering, University of Pennsylvania, Room 311A, Towne Building, 220 South 33rd Street, Philadelphia, PA 19104-6393, USA.

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The smooth flat plate

The velocity distribution

Although the following expressions for the velocity distribution of different segments of the turbulent boundary layer on a smooth flat plate are well known, their basis and derivation are outlined here in order to identify and emphasize the inherent idealizations and related restrictions on their range of applicability.

If the distribution of the time-mean velocity in fully turbulent flow along a smooth plate is postulated to be a function of the shear stress at the wall rather than of the free-stream velocity, and if, near the wall, this distribution is speculated to be independent of the thickness of the boundary layer, and if the variation of the shear stress with distance from the wall is neglected, it follows from dimensional analysis that in this region

$$u^+ = f\{y^+\} \tag{1}$$

Very near the wall, the contribution of turbulence becomes vanishingly small, and Equation 1 reduces to

$$u^+ = y^+ \tag{2}$$

Conversely, if the gradient of the time-mean velocity near the edge of the boundary layer is speculated to be dependent upon the thickness of the boundary layer but to be independent of the viscosity of the fluid, it follows that in this region

$$u_\infty^+ - u^+ = f\left\{\frac{y}{\delta}\right\} \tag{3}$$

If the regions of applicability of Equations 1 and 3 overlap to some extent, it follows that in the region of overlap

$$u^+ = A + B\ln\{y^+\} \tag{4}$$

and

$$u_\infty^+ - u^+ = B\ln\left\{\frac{\delta}{y}\right\} \tag{5}$$

(Equation 4 was also derived by Prandtl (1933) using mixing-length theory. The coefficient B is thereby identified as the reciprocal of the (von Kármán) coefficient of proportionality between the mixing length and the distance from the wall.) The derivation herein of Equations 4 and 5 by speculative and dimensional analysis emphasizes their presumed limitation to the turbulent core but at the same time to the region near enough to the wall so that the variation of the shear stress can be neglected. These expressions, with empirically determined coefficients of $A = 5.0$ and $B = 2.439$ ($k = 0.41$) have been found (see, for example, Coles and Hirst 1969) to represent experimental data well for a fully turbulent boundary layer but only for $30 < y^+ < 0.1\delta^+$. The innermost region, which is not represented by Equations 4 and 5, encompasses the laminar sublayer (represented by Equation 2) and the intervening buffer layer. The outer region, which may be noted to occupy as much as 90 percent of the total boundary layer, is excluded from the domain of Equation 4 by its dependence on the thickness of the boundary layer and on the variation of the shear stress. With this in mind, the expression

$$u_\infty^+ = A + B\ln\{\delta^+\} \tag{6}$$

which is obtained by setting $y^+ = \delta^+$ in Equation 4, is obviously invalid, at least with the same values of A and B as in Equation 4.

A number of expressions have been proposed to interpolate between the domains of Equations 2 and 4 (see, for example, Spalding 1961; Choi and Churchill 1973). However, the improvement in the coefficient of skin friction as a consequence of their use in the derivation that follows proves to be negligible numerically. Hence, this complexity was avoided on practical grounds.

On the other hand, the addition of a function of y/δ to Equation 4, as suggested by Coles (1956) to improve the representation of the velocity distribution in the outer region of turbulent boundary layers, does result in a significant improvement in the subsequently derived expressions for the coefficients of skin friction. For this purpose the particular empirical expression proposed by Hinze (1975) was chosen, resulting in

$$u^+ = A + B\ln\{y^+\} + A'\sin^2\left\{\frac{\pi y}{2\delta}\right\} \tag{7}$$

Since Equation 7 is applicable for the entire turbulent core, it follows that

$$u_\infty^+ = A + A' + B\ln\{\delta^+\} \tag{8}$$

and

$$u_\infty^+ - u^+ = A'\cos^2\left\{\frac{\pi y}{2\delta}\right\} + B\ln\left\{\frac{\delta}{y}\right\} \tag{9}$$

Notation

A	Arbitrary constant in velocity distribution for a smooth surface
A'	Arbitrary coefficient in wake function
B	Arbitrary constant in velocity distribution $= 1/k$
C	Arbitrary constant in velocity distribution for a rough surface
C_f	Local coefficient of skin friction for a plate $= 2\tau_w/\rho u_\infty^2$
C_{fm}	Space-mean coefficient of skin friction for a plate (see Equation 21)
C_{fx^*}	Local coefficient of turbulent skin friction at x^*
$(C_{fmt})_L$	Space-mean coefficient of turbulent skin friction from $x = 0$ to $x = L$
$(C_{fmt})_{x^*}$	Space-mean coefficient of turbulent friction from $x = 0$ to $x = x^*$
$(C_{fml})_{x^*}$	Space-mean coefficient of skin friction for laminar regime from $x = 0$ to $x = x^*$
I_1	Definite integral defined by Equation 14
I_2	Definite integral defined by Equation 15
L	Length of plate (m)
k	Von Kármán constant
Re_L	Overall Reynolds number for a flat plate of length $L = Lu_\infty/\nu$
Re_x	Reynolds number for flat plate at location $x = xu_\infty/\nu$
Re_ε	Reynolds number for uniform roughness on a flat plate $= \varepsilon u_\infty/\nu$
$Re_{\tilde\varepsilon}$	Reynolds number for natural roughness on a flat plate $= \tilde\varepsilon u_\infty/\nu$
u	Local, time-mean velocity (m/s)
u_∞	Free-stream velocity (m/s)
x	Distance along plate from leading edge (m)
x^*	Distance along plate from leading edge to point of transition from a laminar to a turbulent boundary layer (m)
y	Distance from surface of the flat plate (m)
Z	$1/B(2/C_f)^{1/2}$

Greek symbols

δ	Thickness of boundary layer on a flat plate (m)
δ_M	Thickness of boundary layer with respect to momentum (see Equation 19)
ε	Uniform (artificial) roughness (m)
$\tilde\varepsilon$	Effective (natural) roughness (m)
ν	Kinematic viscosity (m^2/s)
ρ	Specific density (kg/m^3)
τ_w	Shear stress on wall (Pa)

Functional notation

s^+	Dimensionless distance $s(\tau_w/\rho)^{1/2}/\nu$ when s is any distance
v^+	Dimensionless velocity $v(\rho/\tau_w)^{1/2}$ when v is any velocity

Equation 9 can also be expressed as

$$\left(\frac{2}{C_f}\right)^{1/2} = A + A' + B \ln\{\delta^+\} \tag{10}$$

where $C_f \equiv 2\tau_w/\rho u_\infty^2$ is the local coefficient of skin friction. The generally accepted value of A' in Equations 7–10 is 2.35. Equations 8 and 9 imply that the velocity attains a value of u_∞ and the velocity gradient a value of zero precisely at $y = \delta$ rather than asymptotically as occurs physically. This subtle idealization is inherent, but unimportant numerically, in the expressions that follow for the coefficients of skin friction.

Dependence of C_f and C_{fm} on Re_x

δ^+, the dimensionless thickness of the boundary layer, is a yet unknown function of the Reynolds number, Re_x. This dependence can be determined from an integral balance of momentum over the boundary layer, namely,

$$\frac{d}{dx}\left(\int_0^\infty \frac{u}{u_\infty}\left(1 - \frac{u}{u_\infty}\right)dy\right) = \frac{\tau_w}{\rho u_\infty^2} \tag{11}$$

The longitudinal variation of the distribution of the dynamic pressure across the boundary layer has been neglected in the formulation of Equation 11, but such an idealization has been shown to be justifiable for all practical purposes.

Equation 11 can be approximated in accordance with the above postulate of the discrete attainment of u_∞ at $y = \delta$ by

$$\frac{d}{dx}\left(\delta \int_0^1 \frac{u}{u_\infty}\left(1 - \frac{u}{u_\infty}\right)d\left(\frac{y}{\delta}\right)\right) = \frac{\tau_w}{\rho u_\infty^2} \tag{12}$$

Equation 13 constitutes the effective definition of δ. In view of the dependence, according to Equation 9, of $u_\infty^+ - u^+$ on y/δ only, and the definition of C_f, Equation 12 can be re-expressed as

$$\frac{d}{dx}\left(\delta B\left(\frac{C_f}{2}\right)^{1/2}\left[I_1 - I_2 B\left(\frac{C_f}{2}\right)^{1/2}\right]\right) = \frac{C_f}{2} \tag{13}$$

where the symbols I_1 and I_2 represent the definite integrals

$$I_1 = \frac{1}{B}\int_0^1 (u_\infty^+ - u^+)\,d\left(\frac{y}{\delta}\right) \tag{14}$$

and

$$I_2 = \frac{1}{B^2}\int_0^1 (u_\infty^+ - u^+)^2\,d\left(\frac{y}{\delta}\right) \tag{15}$$

The substitution of δ from Equation 10 allows integration of Equation 13 from $1/C_f = 0$ at $x = 0$ to obtain an expression that can be arranged as

$$\left(\frac{2}{C_f}\right)^{1/2} = A + A' - B \ln\{2BI_1\} + B \ln\left\{\frac{C_f\left(\mathrm{Re}_x + 2(I_1 + I_2)B^3 \exp\left\{-\frac{A + A'}{B}\right\}\right)}{1 - \left(2 + \frac{I_2}{I_1}\right)B\left(\frac{C_f}{2}\right)^{1/2} + \left(1 + \frac{I_2}{I_1}\right)B^2 C_f}\right\} \tag{16}$$

For the velocity distribution given by Equation 7 with $A = 5.0$, $B = 2.439$, and $A' = 2.35$, evaluation of the definite integrals gives $I_1 = 1.48175$ and $I_2 = 3.8796$. The resulting numerical form of Equation 16 is

$$\left(\frac{2}{C_f}\right)^{1/2} = 2.526 + 2.439 \ln\left\{\frac{C_f(\mathrm{Re}_x + 7.642)}{1 - 7.965C_f^{1/2} + 21.52C_f}\right\} \tag{17}$$

The constant in the numerator of the argument of the logarithm is completely negligible with respect to Re_x in the fully turbulent regime, and Equation 17 can be simplified without significant error to

$$\left(\frac{2}{C_f}\right)^{1/2} = 2.526 + 2.439 \ln\left\{\frac{C_f\,\mathrm{Re}_x}{1 - 7.965C_f^{1/2} + 21.52C_f}\right\} \tag{18}$$

The more complicated form of Equations 16 and 18 relative to their well-known analogs for the friction factor in a round tube, and the appearance of C_f rather than $C_f^{1/2}$ in the numerator of the argument of the logarithm, is wholly a consequence of the growth of the boundary layer in unconfined flow as compared to the invariant one in confined flow. If the wake is neglected completely, Equation 16 is changed only by virtue of $A' = 0$, and Equation 18 additionally only by virtue of slightly different values of I_1 and I_2. The coefficient of 2 has been retained in the left-hand side of Equations 16–18, as well as in subsequent related expressions, in order to preserve the identity of the coefficient B. The reported number of significant figures for I_1 and I_2 is certainly not justified by the functional representation of the wake and the uncertainty of the coefficients used with Equation 7. These values were simply retained for consistency in the evaluation of the coefficients of Equation 18. The four significant figures for the coefficients of Equation 18 are also perhaps excessive on the basis of the form and coefficients of Equation 7; again, they were retained for consistency in subsequent comparisons.

Because of the implicit form of Equation 18 with respect to C_f, analytical integration of this coefficient with respect to x to obtain the space-mean coefficient of skin friction is clearly not feasible. However, the integral term of Equation 11 can be noted to be δ_M, the thickness of the boundary layer with respect to momentum, i.e.,

$$\delta_M \equiv \int_0^\infty \frac{u}{u_\infty}\left(1 - \frac{u}{u_\infty}\right)dy \tag{19}$$

Hence, Equation 11 can be expressed symbolically as

$$\frac{d\delta_M}{dx} = \frac{C_f}{2} \tag{20}$$

from which it follows that

$$\frac{2\delta_M}{L} = \frac{1}{L}\int_0^L C_f\,dx \equiv C_{fm} \tag{21}$$

Equation 21 can be noted to be valid, contrary to Equations 16–18, for any point of transition from a laminar to a turbulent boundary layer and can be recognized in advance as applicable to rough as well as smooth plates.

In view of Equations 19–21 and within the approximation of Equation 11 by Equation 12, it follows from Equations 10 and 13 that

$$\left(\frac{2}{C_f}\right)^{1/2} = A + A' - B \ln\{2BI_1\} + B \ln\left\{\frac{C_{fm}\mathrm{Re}_L}{1 - \frac{I_2}{I_1}B\left(\frac{C_f}{2}\right)^{1/2}}\right\} \tag{22}$$

Equating the right-hand sides of Equations 16 and 22 for $\mathrm{Re}_x = \mathrm{Re}_L$, and again dropping the additive term for Re_x, indicates that

$$C_{fm} = \frac{\left[1 - \frac{I_2}{I_1}B\left(\frac{C_f}{2}\right)^{1/2}\right]C_f}{1 - \left(2 + \frac{I_2}{I_1}\right)B\left(\frac{C_f}{2}\right)^{1/2} + \left(1 + \frac{I_2}{I_1}\right)B^2 C_f} \tag{23}$$

For the values of A, B, and A' corresponding to Equation 18,

$$C_{\mathrm{fm}} = \frac{(1 - 4.516C_{\mathrm{f}}^{1/2})C_{\mathrm{f}}}{1 - 7.965C_{\mathrm{f}}^{1/2} + 21.52C_{\mathrm{f}}} \tag{24}$$

Thus Equation 18 gives C_{f} as a function of Re_x, and Equation 24 in turn gives C_{fm} for $\mathrm{Re}_L = \mathrm{Re}_x$. Because of the substitution of δ from Equation 10, the above expressions for C_{fm} imply initiation of the turbulent boundary layer at the leading edge of the plate. Alleviation of that restriction is undertaken after consideration of rough plates.

An approximate expression for C_{fm} as a direct function of Re_L, analogous to Equation 18 for C_{f} as a function of Re_x, can be derived by combination of Equations 18 and 24 and the truncation of series expansions in recognition of the small value of C_{f} with respect to unity. However, for numerical purposes the calculation of C_{f} from Equation 18 followed by the calculation of C_{fm} from Equation 24 is to be preferred.

Equations 18 and 24 are presumed to be restricted to fully turbulent flows (asymptotically large Reynolds numbers) owing to their derivation from Equation 8 with $A = 5$, $B = 2.439$, and $A' = 2.35$. Before comparing their predictions with experimental data, several prior derivations and correlating equations will be examined.

Prior derivations and correlating equations

Prandtl (1927) apparently derived solutions equivalent to Equations 16 and 23 for $A' = 0$ (no wake), since his calculated values of C_{f} and C_{fm}, as tabulated by Schlichting (1979), p. 642, agree exactly with values obtained from these expressions with $A = 5.56$, $B = 2.54$, and $A' = 0$. Schlichting does not present the expressions themselves because they are "exceedingly inconvenient" (an evaluation that may have been justified before the general availability of electronic computers). Schlichting also correlated these tabulated values with the following purely empirical expressions:

$$C_{\mathrm{f}} = \frac{1}{[2\log_{10}\{\mathrm{Re}_x\} - 0.65]^{2.3}} \tag{25}$$

$$C_{\mathrm{fm}} = \frac{0.455}{[\log_{10}\{\mathrm{Re}_L\}]^{2.58}} \tag{26}$$

Von Kármán (1930) derived the equivalent of Equation 16, using an arbitrary unspecified function of y/δ for the contribution of the wake to the velocity distribution. However, he did not evaluate this general result numerically but instead suggested, as a simplifying assumption, the reduction of the argument of the logarithm to simply $C_{\mathrm{f}}\,\mathrm{Re}_x$. He recognized that some of the omitted terms were probably not negligible, and proposed that the consequent error be compensated for by empirical evaluation of the remaining coefficients. He subsequently (1934) plotted the experimental data of Kempf (1932) for the local coefficient of skin friction in the form of $1/C_{\mathrm{f}}^{1/2}$ versus $C_{\mathrm{f}}\,\mathrm{Re}_x$ in logarithmic coordinates to determine

$$\frac{1}{C_{\mathrm{f}}^{1/2}} = 1.7 + 4.15\log_{10}\{C_{\mathrm{f}}\,\mathrm{Re}_x\} \tag{27}$$

Schoenherr (1932) fitted experimental data for the space-mean coefficient with the following expression of similar form:

$$\frac{1}{C_{\mathrm{fm}}^{1/2}} = \frac{\log_{10}\{C_{\mathrm{fm}}\,\mathrm{Re}_L\}}{0.242} \tag{28}$$

Schultz-Grunow (1940) carried out a derivation similar to that of Prandtl but additionally took into account an experimental wake function. He represented values computed by this process with the following empirical expressions:

$$C_{\mathrm{f}} = \frac{0.370}{(\log_{10}\{\mathrm{Re}_x\})^{2.584}} \tag{29}$$

and

$$C_{\mathrm{fm}} = \frac{0.427}{(\log_{10}\{\mathrm{Re}_L\} - 0.407)^{2.64}} \tag{30}$$

Cebeci and Smith (1974) derived the near equivalent of Equation 16, starting from the point of transition rather than from the leading edge, but thereafter used reduced forms such as Equation 27 for correlation.

Spalding (1962) derived, using his own (1961) inverse relationship for the velocity distribution, the following inverse relationship for the skin friction:

$$\mathrm{Re}_x = \frac{(BZ)^4}{12} + B^3 e^{-A/B} \times \left[e^Z(Z^2 - 4Z + 6) - 6 - 2Z - \frac{Z^4}{12} - \frac{Z^5}{20} - \frac{Z^6}{60} - \frac{z^7}{252}\right] \tag{31}$$

where $Z \equiv 1/B(2/C_{\mathrm{f}})^{1/2}$. The corresponding expression for the mean coefficient of skin friction is

$$\frac{C_{\mathrm{fm}}\mathrm{Re}_L}{2} = \frac{(BZ)^2}{6} + Be^{-A/B} \times \left[e^z\left(1 - \frac{2}{Z}\right) + \frac{2}{Z} + 1 - \frac{Z^2}{6} - \frac{Z^3}{12} - \frac{Z^4}{40} - \frac{Z^5}{180}\right] \tag{32}$$

White (1974), p. 498, correlated values computed from Equation 31 with $B = 2.5$ and $A = 5.5$, while neglecting (justifiably) the two rightmost terms, by the following purely empirical expression:

$$C_{\mathrm{f}} = \frac{0.455}{[\ln\{0.06\,\mathrm{Re}_x\}]^2} \tag{33}$$

He similarly correlated values computed in turn from Equation 32 by the expression

$$C_{\mathrm{fm}} = \frac{0.523}{[\ln\{0.06\,\mathrm{Re}_L\}]^2} \tag{34}$$

White calls Equations 31 and 32 "exact." However, the velocity distribution of Spalding, while conforming to Equations 2 and 4 in the limits of small and large u^+, respectively, and even incorporating a designed functional behavior in the interim, does not account for the wake. This latter failure is demonstrated in Figure 21.3 of Schlichting (1979) as well as in Figure 6-6 of White for a round pipe (for which the wake is less important than for unconfined flow over a flat plate). Equation 31 actually differs from Equation 16 only by virtue of the neglect of the wake ($A' = 0$), the replacement of one negligible term, $2(I_1 + I_2)B^3e^{-((A'+A)/B)}$, by another, $(BZ)^4/12$, and the addition of the six right-most terms (which are almost negligible). However, as demonstrated below, the neglect of the wake does result in a significant error. The differences between Equations 22 and 32 similarly arise only from the term $(BZ)^2/6$ and the six rightmost terms of the latter (which do become appreciable for very small Re_x) and the neglect of the wake, which is again significant.

Comparisons

In Table 1, values of C_f computed from Equations 18, 25, 27, 29, 31, and 33 are compared with one another and with values read from the graphical correlations of experimental data by Smith and Walker (1959) and Spalding and Chi (1964). Values of C_f computed from Equation 16 for $A' = 0$, and hence $I_1 = 1$ and $I_2 = 2$, are included in Table 1 to indicate that specific contribution of the wake. Values computed for Equation 16 for $A' = 0$, $A = 5.5$, and $B = 2.5$ are also included to indicate the effect of the choice of coefficients vis-à-vis Equations 18 and 31.

Equations 18, 25, 27, 29, 31, and 33 all provide reasonable predictions of C_f, particularly for very large Re_x, but Equation 18 appears to provide the best overall representation. The significant effect of the wake is demonstrated by comparison of the predictions of Equations 16 with $A' = 0$, $A = 5$, and $B = 2.439$ and those of Equation 18. The overprediction due to the neglect of the wake varies from 10 percent at $\mathrm{Re}_x = 10^6$ to 6.4 percent at $\mathrm{Re}_x = 10^{10}$. The effect of the choice of the coefficients is similarly demonstrated by the two supplemental calculations for $A' = 0$. The underprediction due to using $A = 5.5$ and $B = 2.5$ instead of $A = 5.0$ and $B = 2.439$ varies from 6.4 percent at $\mathrm{Re}_x = 10^6$ to 5.9 percent at $\mathrm{Re}_x = 10^{10}$. The predictions of Equation 16 with $A' = 0$, $A = 5.5$, and $B = 2.5$ are seen to differ negligibly from those of Equation 31 for all but $\mathrm{Re}_x = 10^5$. This latter agreement confirms the earlier assertion that the improved velocity distribution near the wall vis-à-vis Equation 4 does not affect the prediction of the local coefficient of skin friction significantly. The test calculations also indicate that Equations 25 and 31, which completely neglect the effect of the wake, compensate to a considerable degree by the use of modified values of the coefficients A and B.

As contrasted with all of the other expressions, the coefficients of Equation 27 are based directly on experimental data for the coefficient rather than directly or indirectly on a velocity distribution. Equation 33 is simply an empirical representation for Equation 31.

The predictions of the integrated-mean coefficient of skin friction are similarly compared in Table 2. The remarks relative to the local coefficient are generally applicable here as well. The overprediction due to the neglect of the wake is even greater, decreasing from 13 percent at $\mathrm{Re}_L = 10^6$ to 7 percent at $\mathrm{Re}_L = 10^{10}$. The use of $A = 5.5$ and $B = 2.5$ produces an underprediction of 6.7 percent at $\mathrm{Re}_L = 10^6$ and of 5.9 percent

Table 1 Comparison of expressions for the local coefficient of skin friction on a smooth plate with representative experimental data

Re_x	Exper. data*	$C_f \times 10^3$ Eq. 18	Eq. 16	Eq. 16	Eq. 25	Eq. 27	Eq. 29	Eq. 31	Eq. 33
		$A'=2.35$, $A=5.0$, $B=2.439$	$A'=0$, $A=5.0$, $B=2.439$	$A'=0$, $A=5.5$, $B=2.5$	$A'=0$, $A=5.56$, $B=2.54$			$A'=0$, $A=5.5$, $B=2.5$	$A'=0$, $A=5.5$, $B=2.5$
1×10^5	5.4	5.33	6.08	5.70	5.85	5.77	5.78	5.79	6.01
5×10^5	4.0	3.99	4.48	4.21	4.25	4.17	4.12	4.23	4.28
1×10^6	3.6	3.56	3.97	3.73	3.75	3.67	3.61	3.74	3.76
5×10^6	2.75	2.77	3.05	2.87	2.87	2.80	2.71	2.87	2.86
1×10^7	2.58	2.50	2.74	2.59	2.58	2.51	2.42	2.58	2.57
5×10^7	2.0	2.01	2.19	2.06	2.05	1.985	1.895	2.06	2.05
1×10^8	1.85	1.840	1.994	1.879	1.871	1.807	1.716	1.879	1.868
5×10^8	1.5	1.515	1.630	1.537	1.531	1.473	1.382	1.537	1.535
1×10^9	1.4	1.400	1.502	1.417	1.412	1.356	1.266	1.417	1.418
5×10^9	1.2	1.176	1.255	1.185	1.181	1.131	1.043	1.185	1.194
1×10^{10}	1.1	1.095	1.166	1.101	1.098	1.051	0.964	1.101	1.114

* From graphical correlations of Smith and Walker (1959) and Spalding and Chi (1964).

Table 2 Comparison of expressions for the mean coefficient of skin friction on a smooth plate with representative experimental data

Re_x	Exper. data*	$C_{fm} \times 10^3$ Eq. 24	Eq. 23	Eq. 23	Eq. 26	Eq. 28	Eq. 30	Eq. 32	Eq. 34
		$A'=2.35$, $A=5.0$, $B=2.439$	$A'=0$, $A=5.0$, $B=2.439$	$A'=0$, $A=5.5$, $B=2.5$	$A'=0$, $A=5.56$, $B=2.54$			$A'=0$, $A=5.5$, $B=2.5$	$A'=0$, $A=5.5$, $B=2.5$
1×10^5	6.9	6.70	7.79	7.29	7.16	7.18	7.63	6.66	6.91
5×10^5	4.9	4.90	5.58	5.23	5.10	5.06	5.25	4.89	4.92
1×10^6	4.3	4.32	4.88	4.58	4.47	4.41	4.53	4.34	4.32
5×10^6	3.3	3.30	3.67	3.45	3.37	3.30	3.32	3.35	3.29
1×10^7	2.9	2.96	3.27	3.08	3.00	2.94	2.94	3.02	2.95
5×10^7	2.3	2.34	2.56	2.41	2.35	2.29	2.25	2.39	2.35
1×10^8	2.1	2.13	2.32	2.18	2.13	2.07	2.02	2.17	2.15
5×10^8	1.7	1.728	1.630	1.761	1.715	1.671	1.604	1.757	1.764
1×10^9	1.6	1.589	1.713	1.615	1.571	1.532	1.459	1.612	1.630
5×10^9	—	1.321	1.416	1.336	1.295	1.267	1.187	1.335	1.373
1×10^{10}	—	1.226	1.310	1.237	1.197	1.172	1.091	1.236	1.280

* From graphical correlations of Smith and Walker (1959) and Spalding and Chi (1964).

at $\mathrm{Re}_L = 10^{10}$. Equation 23 with $A' = 0$, $A = 5.5$, and $B = 2.5$ percent deviates somewhat from Equations 32 and 34 owing to the accumulative effect in C_{fm} of significant numerical differences in C_f at very small Re_x. Equations 26 and 32 compensate quite successfully for the neglect of the wake by the use of larger values of the coefficients A and B, but Equation 24 is superior overall.

All in all, the superiority of Equations 18 and 24, whose only empiricism arises from the form and coefficients of Equation 7, relative to the other expressions based on the velocity distribution in predicting values for the local and mean coefficients of skin friction is a consequence of taking into account the wake, while that relative to the purely empirical expressions is a consequence of their more fundamental structure. The improvement over prior expressions is small over the limited range of the available data, but their predictions would be expected to be more reliable for Re_x beyond that range, again because of their theoretical structure.

Rough plates

The process used above to derive expressions for the skin friction on smooth plates might appear to be directly applicable for rough plates, but some intrinsic differences are encountered. The following expression for the velocity distribution in the turbulent boundary layer near a rough plate can be derived by the same reasoning as for a smooth plate:

$$u^+ = C + B \ln\left\{\frac{y}{\varepsilon}\right\} \tag{35}$$

Then, if the same wake-function is postulated for a rough plate, the analog of Equation 7 is

$$u^+ = C + B \ln\left\{\frac{y}{\varepsilon}\right\} + A' \sin^2\left\{\frac{\pi y}{2\delta}\right\} \tag{36}$$

The generally accepted values of C, B, and A' based on experimental velocity distributions are 8.5, 2.439, and 2.35. It follows from Equation 36 that

$$u_\infty^+ = C + A' + B \ln\left\{\frac{\delta}{\varepsilon}\right\} \tag{37}$$

which can also be expressed as

$$\left(\frac{2}{C_f}\right)^{1/2} = C + A' + B \ln\left\{\frac{\delta}{\varepsilon}\right\} \tag{38}$$

Subtraction of Equation 36 and 37 again produces Equation 9, which is thus applicable for both smooth and rough plates.

Dependence of C_f and C_{fm} on x/ε

One of the reviewers of the original version of this paper suggested that the detailed derivation of an expression for the coefficient of skin friction on a rough plate could be avoided by noting that Equations 36–38 differ from Equations 7, 8, and 10 only by virtue of the replacement of A by $C - B \ln\{\varepsilon^+\}$. The analog of Equation 16 is thereby inferred to be

$$\left(\frac{2}{C_f}\right)^{1/2} = C + A' - B \ln\{2BI_1\} + B \ln\left\{\frac{C_f\left(\dfrac{\mathrm{Re}_x}{\varepsilon^+} + (2I_1 + I_2)B^3 \exp\left\{-\dfrac{C + A'}{B}\right\}\right)}{1 - \left(2 + \dfrac{I_2}{I_1}\right)B\left(\dfrac{C_f}{2}\right)^{1/2} + \left(1 + \dfrac{I_2}{I_1}\right)B^2 C_f}\right\} \tag{39}$$

Furthermore, Equations 23 and 24 can then be inferred to be directly applicable for rough as well as smooth plates. Unfortunately, Equation 39 incorporates the consequences of a subtle error, namely, that ε^+, which depends implicitly on τ_w, can be treated as a constant in the integration.

The detailed derivation itself follows a slightly different path than that for a smooth plane, and, in particular, a singularity is encountered at $x = 0$ in one of the terms. Dropping the offending term, which is negligible except very near the landing edge, leads to

$$\left(\frac{2}{C_f}\right)^{1/2} = C + A' - B \ln\{2BI_1\} + B \ln\left\{\frac{C_f\left(\dfrac{\mathrm{Re}_x}{\varepsilon^+} + (2I_2 + I_1)B^2 \exp\left\{-\dfrac{C + A'}{B}\right\}\right)}{1 - \left(2 + \dfrac{I_2}{I_1}\right)B\left(\dfrac{C_f}{2}\right)^{1/2}}\right\} \tag{40}$$

The consequence of the neglect of the variation of ε^+ with x is apparent only in a radical change in the negligible additive term in the numerator and in the added term in the denominator of the rightmost term of Equation 39. This latter term results in significant error because C_f is much larger for rough plates than for smooth ones.

For $C = 8.5$, $B = 2.439$, and $A' = 2.35$, and again $I_1 = 1.48175$ and $I_2 = 3.8769$, Equation 40 becomes

$$\left(\frac{2}{C_f}\right)^{1/2} = 6.026 + 2.439 \ln\left\{\frac{C_f\left(\dfrac{\mathrm{Re}_x}{\varepsilon^+} - 0.476\right)}{1 - 7.965C_f^{1/2}}\right\} \tag{41}$$

The constant term in the numerator of the logarithm is negligible for all practical purposes, permitting reduction of Equation 41 to

$$\left(\frac{2}{C_f}\right)^{1/2} = 6.026 + 2.439 \ln\left\{\frac{C_f \mathrm{Re}_x}{\varepsilon^+(1 - 7.965C_f^{1/2})}\right\} \tag{42}$$

The grouping $C_f \mathrm{Re}_x/\varepsilon^+$ can also be written as $(x/\varepsilon)/(2C_f)^{1/2}$ or as $(2C_f)^{1/2}\,\mathrm{Re}_x/\mathrm{Re}_\varepsilon$ where $\mathrm{Re}_\varepsilon \equiv \varepsilon u_\infty/\nu$. The grouping $(x/\varepsilon)C_f^{1/2}$ is most convenient when considering a fixed location (fixed x/ε) and a varying free-stream velocity (varying C_f), whereas the grouping $(\mathrm{Re}_x/\mathrm{Re}_\varepsilon)/C_f^{1/2}$ is more convenient for consideration of a fixed free-stream velocity (fixed Re_ε) and a varying distance along the plate (varying $\mathrm{Re}_x C_f^{1/2}$).

The detailed derivation of an expression for C_{fm} for a rough plate does follow the same path as for a smooth plate and results in Equations 23 and 24 except for the term in C_f in the denominator, owing to its absence from Equation 40. Thus the numerical counterpart of Equation 42 is

$$C_{fm} = \frac{(1 - 4.516C_f^{1/2})C_f}{1 - 7.965C_f^{1/2}} \tag{43}$$

Extension for natural roughness and a full range of Re_x

Equation 42 is implied to be restricted to asymptotically large values of Re_x such that C_f is a function only of the roughness. It can, however, be combined with Equation 18 by analogy to the combination of friction factors for smooth and rough pipes by Colebrook (see Churchill 1973) to construct the following expression for all Re_x in the fully turbulent regime for an effective natural roughness $\tilde{\varepsilon}$:

$$\left(\frac{2}{C_f}\right)^{1/2} = 2.525 + 2.439 \ln \times \left\{\frac{C_f \mathrm{Re}_x}{(1 - 7.965 C_f^{1/2})(1 + 0.238\tilde{\varepsilon}^+) + 21.52 C_f}\right\} \quad (44)$$

Since $\tilde{\varepsilon}^+$ varies with Re_x owing to the change in $(\tau_w \rho)^{1/2}$, the replacement of $\tilde{\varepsilon}^+$ by $\mathrm{Re}_{\tilde{\varepsilon}}\,(C_f/2)^{1/2}$ is again convenient.

Prior derivations and correlating equations

Von Kármán (1930, 1934) derived an approximate expression for plates with a uniform artificial roughness and fitted the coefficients using expressions for the friction factor of artificially roughened pipe to obtain

$$\frac{1}{C_f^{1/2}} = 5.8 + 4.5 \log_{10}\left\{\frac{x}{\varepsilon} C_f^{1/2}\right\} \quad (45)$$

According to Schlichting (1979), p. 652, Prandtl and Schlichting (1934) determined C_f and C_{fm} for uniformly roughened plates by the same procedure as used by Prandtl (1927) for smooth plates. Their solution appears to have been equivalent to Equations 42 and 43 for $C = 8.5$, $B = 2.56$, and $A' = 0$. They fitted their numerically computed values of C_f and C_{fm} with

$$C_f = \frac{1}{\left(2.87 + 1.58 \log_{10}\left\{\frac{x}{\varepsilon}\right\}\right)^{2.5}} \quad (46)$$

and

$$C_{fm} = \frac{1}{\left(1.89 + 1.62 \log_{10}\left\{\frac{L}{\varepsilon}\right\}\right)^{2.5}} \quad (47)$$

White (1974), pp. 503–504, utilized a velocity distribution that can be generalized slightly as

$$u^+ = A + B \ln\left\{\frac{y^+}{1 + \tilde{\varepsilon}^+ e^{-(C - A/B)}}\right\} \quad (48)$$

to derive the equivalent of

$$\mathrm{Re}_x = B^3[Z^2 - 4Z + 6 + \tilde{\varepsilon}^+(Z^2 - 5Z + 7)e^{-(C - A/B)}]e^{(Z - (A/B))} \quad (49)$$

and

$$C_{fm} = \frac{2B}{\mathrm{Re}_L}\left[1 + \mathrm{Re}_{\tilde{\varepsilon}}\left(\frac{C_f}{2}\right)^{1/2} e^{-(C - A/B)}\right]\left[1 - 5\left(\frac{C_f}{2}\right)^{1/2}\right]e^{(Z - (A/B))} \quad (50)$$

For very large roughness (for which ε and $\tilde{\varepsilon}$ do not need to be distinguished), Equations 49 and 50 reduce to

$$\frac{x}{\varepsilon} = B^2\left(Z - 5 + \frac{7}{Z}\right)e^{(Z - (C/B))} \quad (51)$$

and

$$C_{fm} = 2B\,\frac{\varepsilon}{L}\left(\left(\frac{C_f}{2}\right)^{1/2} - \frac{5C_f}{2}\right)e^{(Z - (C/B))} \quad (52)$$

for which White proposed the following explicit empirical representations:

$$C_f = \frac{1}{\left[1.4 + 3.7 \log\left\{\frac{x}{\varepsilon}\right\}\right]^2} \quad (53)$$

and

$$C_{fm} = 0.024\left(\frac{\varepsilon}{L}\right)^{1/6} \quad (54)$$

Equations 49, 51, and 52 differ from Equations 44, 40, and 43 directly or indirectly only owing to the neglect of the wake in Equation 48. On the other hand, the coefficients recommended by White, namely, $A = 5.5$, $B = 2.5$, and effectively $C = 8.5$, result in a further, somewhat compensatory difference just as for smooth plates.

Mills and Hang (1983) integrated numerically the differential model that led to Equation 39, using $C = 8.5$, $B = 2.439$, and $A' = 2.682$. They avoided the singularity encountered in the analytical solution by starting at unspecified finite values of x and C_f. Their computed values of C_f, which were not given, are said to be represented within 1 percent for $150 < (x/\varepsilon) < 1.5 \times 10^7$ by

$$C_f = \frac{1}{\left(3.476 + 0.707 \ln\left\{\frac{x}{\varepsilon}\right\}\right)^{2.46}} \quad (55)$$

They integrated their computed values of C_f numerically to obtain C_{fm}. These values were in turn fitted to the same asserted accuracy as for Equation 55 by

$$C_{fm} = \frac{1}{\left(2.635 + 0.618 \ln\left\{\frac{L}{\varepsilon}\right\}\right)^{2.57}} \quad (56)$$

Comparisons

Experimental data for the local coefficient of skin friction on rough plates are rather limited. The values determined by Pimenta et al. (1979) for a uniformly roughened plate are utilized in Table 3 as a basis of comparison for values predicted by Equations 42, 45, 46, 51 (with $C = 8.5$ and $B = 2.5$), 53, and 55. Values predicted by Equation 40, with $C = 8.5$, $B = 2.439$, and $A' = 0$ and with $C = 8.5$, $B = 2.5$, and $A' = 0$, are included to define the effect of neglecting the wake and of the choice of the numerical value for the coefficient B. The differences between these predictions are somewhat greater than those of Table 1 for the smooth plate. Equation 42 is in the best agreement with this particular set of experimental data, but the differences between Equations 55 and 42 are not much greater than the scatter of the data. These latter small differences are the net consequence of the use of slightly different values of A', numerical versus approximate analytical integration, and empirical representation of the values calculated by numerical integration. The value of $A = 2.68$ used in deriving Equation 55 was actually inferred from the velocity distributions measured by Pimenta et al. in conjunction with their measurements of C_f.

Equation 45, which is a purely empirical expression inferred from experimental data for uniformly roughened pipes is also in fair agreement with the measured values of C_f in Table 3. The test calculations with Equation 40 indicate that neglecting the wake results in an overprediction of C_f decreasing from 22 percent at $(x/\varepsilon) = 100$ to 15 percent at $(x/\varepsilon) = 3000$, whereas increasing the coefficient B from $1/0.41 = 2.439$ to $1/0.4 = 2.5$ reduces the prediction of C_f by less than about 2 percent for all x/ε. This explains the significant overpredictions by Equations 46 and 51; adjustment of B alone is inadequate to compensate for the neglect of the wake. The small differences between the predictions of Equation 40 with $A' = 0$ and $B = 2.5$ and Equation 51 are presumably due to the improved velocity

Table 3 Comparison of expressions for the local coefficient of skin friction on a uniformly roughened plate with experimental data of Pimenta et al. (1979)

x/ε	$C_f \times 10^5$								
	Exper.	Eq. 42	Eq. 40	Eq. 40	Eq. 45	Eq. 46	Eq. 51	Eq. 53	Eq. 55
		$A'=2.35$ $C=8.5$ $B=2.439$	$A'=0$ $C=8.5$ $B=2.439$	$A'=0$ $C=8.5$ $B=2.5$		$A'=0$ $C=8.5$ $B=2.56$	$A'=0$ $C=8.5$ $B=2.5$	$A'=0$ $C=8.5$ $B=2.5$	$A'=2.68$ $C=8.5$ $B=2.43$
100	—	837	1017	998	1009	1120	1031	1291	918
250	—	696	830	814	774	874	824	948	732
500	—	606	714	699	645	736	701	771	625
758	534	585	653	639	582	667	639	688	571
838	522	548	639	626	568	652	625	670	559
1145	504	516	599	586	528	607	584	618	524
1226	504	509	590	578	519	598	576	608	517
1532	478	488	564	552	494	570	549	575	494
1613	486	483	558	546	488	563	543	568	489
1919	462	467	539	527	470	543	523	545	472
2000	472	464	534	522	465	538	519	540	468
2306	452	452	519	508	451	522	504	522	455
2387	458	449	516	504	448	518	501	518	452
2694	444	439	503	492	436	505	489	504	442
2774	448	437	501	489	434	502	486	500	440
3000	—	430	493	482	427	494	478	491	433

Table 4 Comparison of expressions for the local coefficient of skin friction on a uniformly roughened plate with experimental data of Pimenta et al. (1979)

L/ε	$C_f \times 10^5$						
	Eq. 43 & 42	Eq. 43 & 40	Eq. 43 & 40	Eq. 47	Eq. 52 & 51	Eq. 54	Eq. 56
	$A'=2.35$ $C=8.5$ $B=2.439$	$A'=0$ $C=8.5$ $B=2.439$	$A'=0$ $C=8.5$ $B=2.5$	$A'=0$ $C=8.5$ $B=2.56$	$(A'=0)$ $C=8.5$ $B=2.5$	$A'=0$ $C=8.5$ $B=2.5$	$A'=2.68$ $C=8.5$ $B=2.439$
100	1812	2181	2197	1678	2013	1114	1262
250	1294	1533	1532	1248	1480	956	980
500	1035	1212	1205	1019	1194	852	822
750	917	1067	1059	912	1061	796	746
1000	845	979	970	845	978	759	697
1250	795	918	908	798	920	731	663
1500	757	872	862	762	876	709	636
1750	727	836	825	733	841	691	615
2000	702	806	795	710	812	676	597
2250	681	781	770	690	787	663	583
2500	664	759	749	673	767	651	570
2750	648	741	730	658	749	641	558
3000	634	724	714	644	733	632	548

distribution near the wall in the latter. The differences between Equation 40 with $A' = 0$ and $B = 2.5$ and Equation 46 are presumably due to errors in the empirical representation of computed values by the latter, rather than to the slightly different value of B. Equation 53, which is purported to be an empirical, explicit representation for Equation 51, is in considerable error in that respect for small (x/ε), differing by 25 percent at $(x/\varepsilon) = 100$.

Values of C_{fm} predicted for rough plates by Equation 43 (with C_f from Equation 42), Equation 47, Equation 52 (with C_f from Equation 51), Equation 54, and Equation 56 are compared in Table 4. Values calculated from Equation 43 (with C_f from Equation 40) for $C = 8.5$, $A' = 0$, and $B = 2.439$ and for $C = 8.5$, $A' = 0$, and $B = 2.5$ are included to define the effects of neglecting the wake and of the choice of a numerical value for the coefficient B. Unfortunately, a reliable set of experimental data to serve as a criterion for evaluation of the various expressions could not be identified. Several conclusions are nevertheless possible concerning the predictions of C_{fm} for rough plates. Elimination of the term representing the wake leads to an overprediction of C_{fm} ranging from 20 percent at $(L/\varepsilon) = 100$ to 14 percent at $(L/\varepsilon) = 3000$, while increasing B from 2.439 to 2.5 results in an underprediction of less than 1.5 percent for any (x/ε). The differences between the predictions of Equation 52 (with Equation 51) and Equation 43 (with Equation 40) for the same values of A', C, and B presumably reflect the effect of the improved velocity distribution near the wall in the former pair. The large differences between the predictions of Equation 47 and Equation 43 (with Equation 40) are presumably due to error in the empirical representation by the former, rather than to the slightly different values of B. The much lower predictions of Equation 56 relative to Equation 43 (with Equation 42) cannot be explained on the basis of the slightly different values of A' and may be due to the use of a finite starting value of x in the former. The very poor empirical representation of the values computed from Equation 52 (with Equation 51) by Equation 54 is inexplicable and may involve an error of transcription.

Although experimental data are not available to test Equations 44 and 49, the former is recommended on the basis of the best predictions of C_f for asymptotically large ε/x. Equation 43 combined with Equation 44 is similarly recommended over Equation 50 on the basis of the incorporation of a term for the effect of the wake.

Effect of transition

The above expressions for C_{fm} (except possibly Equation 56) all postulate the onset of a turbulent boundary layer at the leading edge of the plate. If the boundary layer is completely laminar up to some critical length x^* and fully turbulent thereafter, a mean coefficient can be estimated from

$$C_{fm} = (C_{fmt})_L - \frac{x^*}{L}(C_{fmt} - C_{fml})_{x^*} \qquad (57)$$

For a naturally rough plate, Equation 44 would appear to be the most reliable equation to use for C_{fmt}, since it is applicable for the regime of developing turbulence due to the roughness.

Summary and conclusions

Expressions in closed form have been derived for the local and mean coefficients of skin friction in a fully turbulent boundary layer on smooth and rough plates. These expressions are based on the generally accepted semitheoretical expressions for the velocity distribution, including the wake. The only further approximations arise from the implicit determination of the thickness of the boundary layer from an integral momentum balance, and from the postulate that the turbulent boundary layer is initiated at the leading edge of the plate. This latter approximation can be relaxed for C_{fm} by combining the indicated expressions with that for the laminar segment of the boundary layer per Equation 57.

The predictions of the expressions derived herein agree with the best available experimental data for the coefficients of skin friction within their uncertainty, and appear to be more successful in this respect than any of the current semitheoretical and empirical expressions, including, remarkably, those that have been fitted to these very data. Accordingly, the expressions derived herein are presumed to be the most reliable for extrapolation to higher Re_x and to a wider range of roughness than encompassed by the data.

More accurate and extended experimental data or numerical solutions may in the future suggest minor modifications in the coefficients of the semitheoretical expressions for the velocity distribution and hence for the expressions derived herein for the coefficients of skin friction, but are unlikely to improve upon the form of the latter expressions.

The expressions derived herein for the coefficients of skin friction are implicit in form. That is, they must be solved iteratively for a specified value of Re_x and/or of the roughness ratio ε/x (or ε^+ or Re_ε), but such calculations are feasible and rapidly convergent even with a hand-held computer.

The other characteristics of the turbulent boundary layer, such as the nominal thickness, the displacement thickness, the momentum thickness, and the variation of the shear stress with distance from the wall, can readily be derived from the expressions developed herein for the skin friction.

Early workers, including von Kármán, Prandtl, Schlichting, and Schultz-Grunow, recognized the possibility of deriving expressions such as those obtained herein for the coefficients of skin friction, but, in the absence of electronic computers, were deterred by the complexity and implicit character of these expressions. More recently, White presented detailed implicit expressions that are similar to those herein except for the neglect of the wake. He nonetheless derived and recommended empirical approximations.

The odd-valued exponents of the prior correlating equations are a consequence of their simplistic structure vis-à-vis those derived herein.

Further details concerning the derivations will appear in a forthcoming book (Churchill in preparation).

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References

Cebeci, T. and Smith, A. M. O. 1974. *Analysis of Turbulent Boundary Layers*. Academic Press, New York

Churchill, S. W. 1973. Empirical expressions for the shear stress in turbulent flow in a commercial pipe. *AIChE J.*, **19**, 395–396

Churchill, S. W. (in preparation). *Turbulent Flows. The Practical Use of Theory.*

Churchill, S. W. and Choi, B. 1973. A simple expression for the velocity distribution in turbulent flow in smooth pipes. *AIChE J.*, **19**, 196–197

Coles, D. 1956. The law of the wake in the turbulent boundary layer. *J. Fluid Mech.*, **1**, 191–226

Coles, D. E. and Hirst, E. A. (eds.). 1969. *Proceedings, Computations of Turbulent Boundary Layers— 1968, AFOSR-IFP-Stanford Conference, Vol. 2, Compiled Data*. Thermodynamics Division, Department of Mechanical Engineering, Stanford University, Stanford, CA

Hinze, J. O. 1975. *Turbulence*, 2nd ed. McGraw-Hill, New York

Kempf, G. 1932. Reibungsergebnisse an ebenen glatten und rauhen Flächen. *Hydrodynamische Probleme des Schiffsantriebs*, **1**, 74–82

Mills, A. F. and Hang, X. 1983. On the skin friction coefficient for a fully rough flat plate. *J. Fluids Eng. Trans. ASME*, **105**, 364–365

Pimenta, M. M., Moffat, R. J., and Kays, W. B. 1979. The structure of a boundary layer on a rough wall with blowing and heat transfer. *J. Heat Transfer Trans. ASME*, **101**, 193–198

Prandtl, L. 1927. Über den Reibungwiderstand strömunder Luft. *Ergebnisse Aerodyn. Versuchsanst zu Göttingen*, **3**, 3–5

Prandtl, L. 1933. Neuere Ergebnisse der Turbulenzforschung. *Z. Ver. Deut. Ing.*, **77**, 105–114

Prandtl, L. and Schlichting, H. 1934. Das Widerstandsgesetz rauher Platten. *Werft*, Reederei, Hafen, 1–4

Schlichting, H. 1979. *Boundary Layer Theory*, English transl. by J. Kestin, 7th ed. McGraw-Hill, New York

Schoenherr, K. E. 1932. Resistance of flat surfaces moving through a fluid. *Trans. Soc. Nav. Arch. Marine Engrs.*, **40**, 279–313

Schultz-Grunow, F. 1940. Neues Widerstandsgesetz für glatten Platten, *Luftfahrtforschung*, **17**, 239 (1941. English transl., New frictional resistance law for smooth plates. *NACA* TM986, Washington, DC

Smith, D. W. and Walker, J. D. 1959. Skin friction measurements in incompressible flow. *NASA* TR R-26, Washington, DC

Spalding, D. B. 1961. A single formula for the "Law of the Wall." *J. Appl. Mech.*, **28**, 455–458

Spalding, D. B. 1962. A new analytical expression for the drag of a flat plate valid for both the turbulent and laminar regimes. *Int. J. Heat Mass Transfer*, **5**, 1133–1138

Spalding, D. B. and Chi, S. W. 1964. The drag of a compressible turbulent boundary layer on a smooth flat plate with and without heat transfer. *J. Fluid Mech.*, **18**, 117–143

von Kármán, T. 1930. Mechanische Ähnlichkeiten und Turbulenz. *Proc. 3rd Int. Congr. Appl. Mech., Stockholm*, **1**, 90–92

von Kármán, T. 1934. Turbulence and skin friction. *J. Aeronaut Sci.*, **1**, 1–20

White, R. M. 1974. *Viscous Fluid Flow*. McGraw-Hill, New York